CHAPTER 3

STATISTICAL DISTRIBUTIONS

CONTENTS

1 INTRODUCTION

1.1 R&M parameters are statistical in nature. Although they are very much equipment performance parameters they address the performance over a long period. Actual failures will occur at random and repair times will show a spread. Understanding the distribution of these random events and variations is important in engineering practical solutions to operational problems. This requires an understanding of the relevant statistical distributions.

1.2 This Chapter introduces the statistical concepts, definitions and distributions that are most relevant to R&M activities. Where appropriate it comments on the aspects of R&M to which specific distributions apply.

1.3 The application to real world data is not considered here. Activities such as distribution Paramenter Estimation, Confidence Intervals, Goodness of Fit Testing and Hypothesis Testing provide such a connection and are addressed in Pt4 Ch11.

2 DEFINITIONS AND CONCEPTS

2.1 Random Variable, Population and Sample

2.1.1 The subject of statistics is concerned with the characteristics of a set of things or events. The complete set is termed the **population**. The parameter which is chosen to characterise or describe the individual elements of the set is called the **random variable** (designated 'x' hereafter). Normally it will not be possible to measure x for the every element of the set when it is desired to characterise a population. It is therefore common to estimate the characteristics of a population by measuring the x values of a **sample**.

2.1.2 There are two types of random variable: a **continuous** random variable and a **discrete** random variable. Continuous variables can take any value between the lower and upper limits (all real values of x in the range are possible, x is 'infinitely dense'). Measurements such as length, duration, current, etc are continuous variables. Discrete variables can only take specific values (normally integers). Examples include the number of goals scored in a football match or the value of cash carried by a person.

2.1.3 It is important that samples be representative of the whole population. Suppose, for example, one were interested in the height of adult males in the UK. In this case, the random variable is 'height'. One would not need to measure the whole population to obtain a good estimate of, for instance, the average height - a sample would be sufficient. However, the sample would have to be drawn from all over the UK to allow for regional differences and from carefully chosen age, ethnic and social samples. In a similar way, it is necessary, when measuring the attributes of a system or type of equipment, to ensure that the sample is of sufficient size and distribution to be adequately representative of the population.

2.2 Concept of Distribution

2.2.1 As indicated by the name, a random variable may generally take values over a range; i.e. it is **distributed** over the range. In some parts of the range the variable occurs more frequently than in others. That is, its probability of occurrence varies over the range. The manner in which this probability varies over the range characterises the **distribution** of the random variable.

2.3 PDF and CDF for continuous random variables

2.3.1 Two functions are used to describe a distribution: its **Probability Density Function** (PDF) and its **Cumulative Distribution Function** (CDF). They are different forms of the same function.

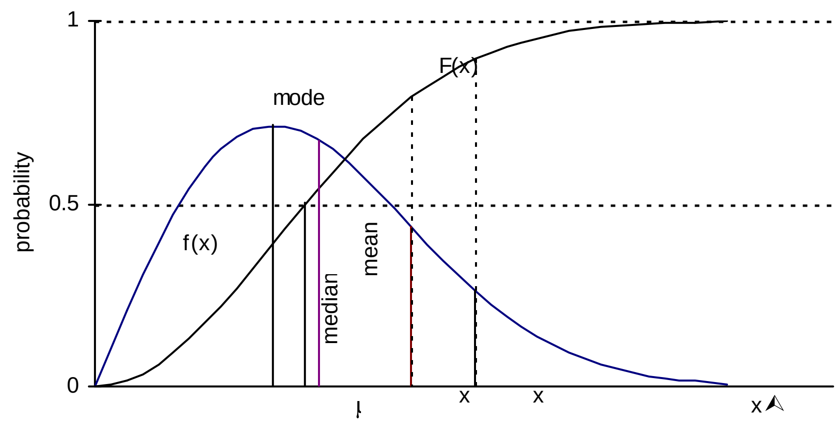

Figure 1: PDF [f(x)] and CDF [F(x)] for a continuous variable

2.3.2 Figure 1 shows the PDF and CDF for a **continuous** random variable x. The usual notation for these functions is f(x) and F(x) respectively. The PDF is defined such that the area under the PDF between x_1 and x_2 is equal to the probability that a random observation x lies in the interval x_1 to x_2. Put mathematically:

$$P_{(\text{x lies between } x_1 \text{ and } x_2)} = \int_{x_1}^{x_2} f(x)dx$$

2.3.3 Suppose that the population range of the random variable is x_L to x_U (negative infinity, zero, positive infinity or any other value is valid). It is certain (probability = 1) that x lies between x_L and x_U. Therefore the area under f(x) between these limits is equal to unity:

$$\int_{x_L}^{x_U} f(x)dx = 1$$

2.3.4 A common error made concerning PDFs is the notion that f(x) is the probability of the exact value x occurring. This is *not* true for a continuous variable; it is necessary to associate the PDF with an interval to obtain a probability of occurrence. The probability of any particular value of x occurring is zero when x ranges over a continuum because there are an infinite number of possible values of x. For continuous distributions, f(x) is not a probability at all and may exceed 1.

2.3.5 A parameter which is frequently required is the probability that an observation will be less (or greater) than some specified value, x_i say. From the definition of f(x) above, it can be seen that this probability is the area under f(x) to the left (or right) of x_i. Put mathematically:

$$P_{(x<x_i)} = \int_{x_L}^{x_i} f(x)dx$$

$$P_{(x>x_i)} = \int_{x_i}^{x_U} f(x)dx$$

2.3.6 $P(x<x_i)$ is called the **cumulative distribution function** and is generally designated F(x). It is shown in Figure 1 related to its equivalent PDF. F(x) increases from 0 to 1 as x increases from x_L to x_U.

2.4 PDF and CDF for discrete random variables

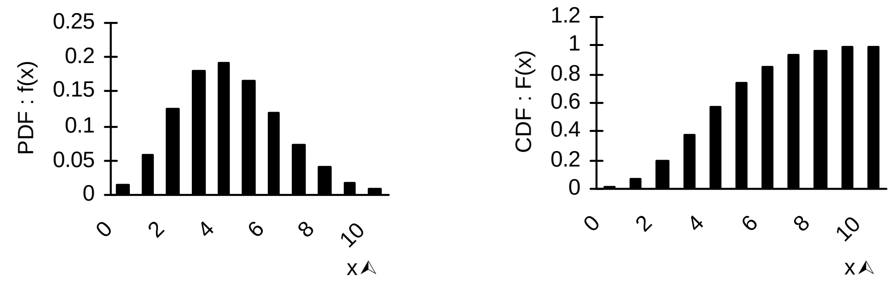

Figure 2: PDF [f(x)] and CDF [F(x)] for a discrete variable

2.4.1 The concepts of PDF and CDF apply to discrete random variables as well as continuous random variables. Figure 2 shows a discrete distribution. Each valid x value has a value for f(x) and F(x) but the values in between do not. Unlike distributions of continuous random variables, for discrete random variables it is true to say that f(x) represents the probability of that specific value occurring.

2.4.2 It is also true to say that the sum of f(x) for all valid values of x is unity.

$$\sum_{-\infty}^{\infty} f(x)dx = 1$$

For most discrete variables x cannot be negative. Therefore zero can normally be substituted for minus infinity in the equation.

2.5 Mean, Median, Mode, Variance and Quantiles

2.5.1 The **mean** value (□) of a distribution is the arithmetic average of the population variable. For a population of size N:

$$\mu = \frac{1}{N}\sum_{1}^{N} x_j$$

For a distribution with PDF = f(x):

$$\mu = \sum_{x_L}^{x_U} x.\,f(x).\,dx$$

A mechanical analogy is that the mean occurs at the 'centre of gravity' along the x axis of the area under the PDF. The population mean is often designated by μ, and a sample mean by $\overline{x}$.

2.5.2 The **median** value of a distribution is that value of x for which 50% of the population is higher, 50% lower. It is often designated x_m. Note that:

$$F(x_m) = 0.5$$

i.e. the areas either side of x_m in the PDF figure are equal.

2.5.3 A **mode** of a distribution is a peak value of the PDF. In general, it indicates a local region of the range where observations occur most frequently. A distribution may have more than one mode, but this is not the case for the more common analytical distributions introduced in this Chapter.

2.5.4 The mean, median and mode only coincide when the PDF is symmetrical and unimodal, e.g. the Normal Distribution. They will not coincide when the distribution is skewed.

2.5.5 The **variance** of a distribution measures the degree of dispersion of the variable about the mean. When the variance is small, the PDF is tall and thin; when large, the PDF is low and wide. Note that variance is not really associated with range; for example, the Normal Distribution has a *range* from $-\infty$ to $+\infty$, but the *variance* of a particular Normal Distribution could take any value. Variance (denoted by □^{2}) is computed as:

$$\sigma^2 = \frac{1}{N}\sum_{1}^{N}(x_j - \mu)^2$$

for a Population of N things with mean μ. That is, it is the average value of the square of the deviations from the mean. In terms of PDF:

$$\sigma^2 = \sum_{x_L}^{x_U}(x_j - \mu)^2.\,f(x).\,dx$$

2.5.6 An alternative measure of dispersion is the **standard deviation** (denoted by □) which is the square root of variance.

2.5.7 The y **quantile** (or **fractile**) is the value of the random variable for which the cumulative distribution function equals y ($0 \le y \le 1$) or "jumps" from a value less than y to a value greater than y. If the cumulative distribution function equals y throughout an interval between two consecutive values of the random variable, then any value in this interval may be consider as the p quartile. The y quartile is often designated x_y.

$$F(x_y) = y/100$$

i.e. y% of the area in the PDF figure lies to the left of x_y.

x_{50} is the median, x_{25} the **lower quartile**, x_{75} the **upper quartile**, x_{90} the 90% **decile**, etc.

2.5.8 A **percentile** is defined in a corresponding manner to a quantile but with y expressed as a percentage. This leads to the common designations:

- x_{50} for the median
- x_{25} the **lower quartile**
- x_{75} the **lower quartile**

2.5.9 Quantiles and percentiles are often used in defining a maximum parameter. For example when defining the acceptable repair time, the maximum time is often relevant. However there is no true maximum to the distribution as the tail of the PDF tends asymptotically to zero. Specifying the 90th or 95th percentile provides a good solution.

3 BINOMIAL DISTRIBUTION

3.1 Nature

3.1.1 The Binomial Distribution is a discrete distribution that describes the distribution of the number of successes in n independent trials, where the probability of success at each trial is p.

For n = 1 the probability of 1 success is p
and the probability of failure (q) is 1-p

For n = 2 the probability of 2 successes is p^2
the probability of 1 success and 1 failure is 2p.q
and the probability of 2 failures q^2

In general the probability of x successes in n trials is ${}_nC_x.p^x.q^{(n-x)}$

3.1.2 The PDF, f(x), is the probability of obtaining exactly x successes in n trials ($0 \leq x \leq n$, x and n integer).

3.1.3 The CDF, F(x), is the probability of obtaining x or less (no more than x) successes in n trials ($0 \leq x \leq n$, x and n integer).

3.2 Application

3.2.1 The Binomial Distribution is generally applicable where the a system is put through a number of independent trials, provided that the outcome of each trial is success or failure and the probability of success is constant. It is fundamental to many types of acceptance sampling scheme in which the decision to accept or reject a batch is made on the basis of the number of defective items in a sample.

3.2.2 A simple example is the tossing of a coin 10 times. The probability of it landing head up on each individual trial (p) is 0.5. The probability of it landing head up every time is p^n ($0.5^{10} \approx 0.001$). The probability of it landing tail up once and head up 9 times is ${}_{10}C_9.p^9.q^1$ (${}_{10}C_9.0.5^9.0.5^1 \approx 0.0097$), etc. Exactly the same figures would apply to the firing of 10 missiles and their exploding on target, if the Reliability of each missile to reach the target and explode is 0.5.

3.2.3 The situation is similar when maintaining equipment. If the 90% decile of maintenance task times for an item is 1 hour, then in any random sample of 5 tasks, the probability of all 5 tasks being completed in 1 hour each is ${}_5C_5.0.9^5.0.1^0$ (= 0.59). The probability that 4 tasks are each completed in 1 hour and 1 takes longer is ${}_5C_4.0.9^4.0.1^1$ (= 0.33), etc.

3.3 Details

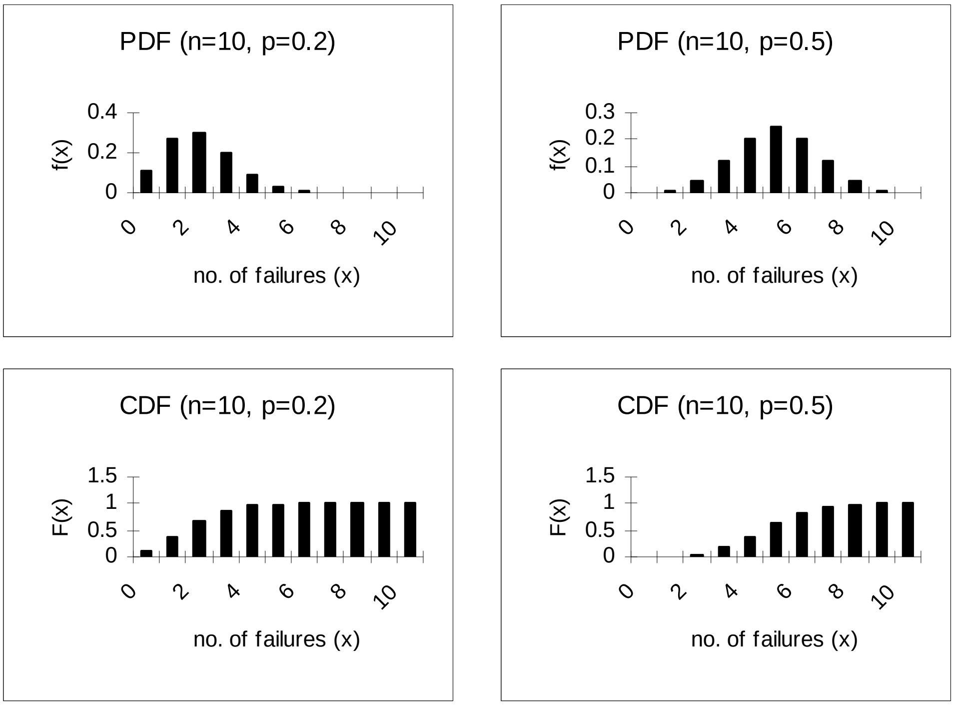

Figure 3: PDF and CDF for the Binomial Distribution

Note: $_{n}C_{i} = \dfrac{n!}{i!.(n-i)!}$

$n! = n.(n-1).(n-2)\ldots 2.1$

$0! = 1$

PDF: $f(x) = {}_{n}C_{x}\, p^{x}\, q^{(n\text{-}x)}$

CDF: $F(x) = \sum_{0}^{x} {}_{n}C_{i}\, p_{i}\, q^{(n-i)}$

mean: $\mu = n.p$

mode: x where $f(x\text{-}1) \le f(x) \ge f(x+1)$

median: x where $F(x) = 0.5$

variance: $\sigma^2 = n.p.q$

standard deviation: $\sigma = \sqrt{n.p.q}$

For n.p and n.q both > 5, the Binomial distribution approximates to a Normal Distribution with $\mu = n.p$ and $\sigma = \sqrt{n.p.q}$.

4 NEGATIVE EXPONENTIAL DISTRIBUTION

4.1 Nature

4.1.1 Negative exponential decay describes the natural decay of a parameter. It is based on the constant e (the base of natural logarithms) and a drain on that parameter at rate that is a constant proportion of the value of that parameter.

4.1.2 This decay becomes a distribution when applied to parameters, such as reliability, and the standard distribution descriptors can be applied.

4.2 Application

4.2.1 The voltage on a capacitor (of capacitance C) connected across a resistor (of resistance R) decays exponentially. If the voltage is V_0 at time t_0 then the Voltage V_1 at time t_1 is given by:

$$V_1 = V_0 e^{\frac{-(t_1 - t_0)}{RC}}$$

4.2.2 Reliability (that is the probability of equipment not having failed) behaves in a similar way with respect to time (when the failure rate is constant):

$$R = e^{-\lambda t}$$

where λ is the constant failure rate and t is the time since the last time at which the equipment was known not to be in a failed state (R=1 when t=0). The Reliability of an item of equipment decays exponentially with time until it is tested. Providing that the test is satisfactory, the Reliability returns to 1 and the decay begins again. If a number of trials are performed where such items are left for various periods of time and then tested then the distribution of the results will follow the negative exponential distribution.

4.3 Details

PDF: $f(x) = \lambda e^{-\lambda x}$ (λ is a constant >0, $x \geq 0$)

CDF: $F(x) = 1 - e^{-\lambda x}$

mean: $\mu = 1/\lambda$ median: $x = (\log_e 2)/\lambda$

mode: 0

variance: $1/\lambda^2$ standard deviation: $\sigma = 1/\lambda$

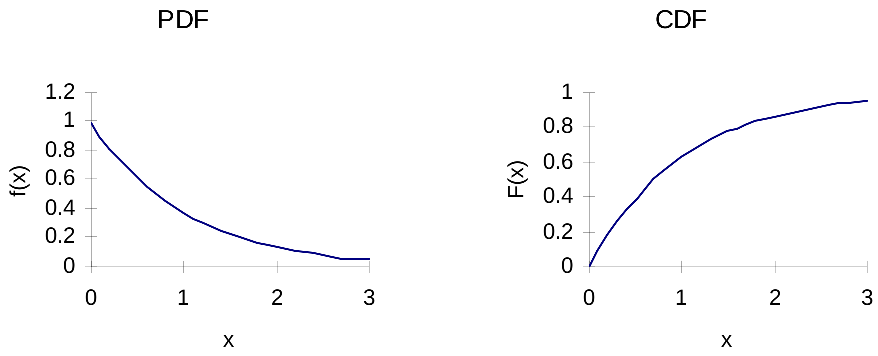

Figure 4: PDF and CDF for the negative exponential distribution

5 NORMAL DISTRIBUTION

5.1 Nature

5.1.1 The Normal* distribution is a continuous distribution given by the equation for its PDF (see subsection 5.3 for an explanation of the symbols):

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

5.1.2 With a continuous distribution, it is the area under the curve that is meaningful. However there is no simple formula for the CDF. It is possible to convert the PDF expression to a series, integrate each significant term and sum the results (feasible using modern computer-based spreadsheets) but not necessary. The same modern computer spreadsheets generally offer functions which return the value of the CDF for a given value of x.

5.1.3 The CDF is commonly provided in tables based on the **standard normal distribution.** This is a normal distribution with a mean of zero and standard deviation of unity (see Table 1). For any normally distributed variable, x, the transformation $z = \frac{x-\mu}{\sigma}$ yields a variable, z, which relates to the standard normal distribution. z is called the **standard normal deviate**.

5.1.4 Alternatively, an approximation to the CDF, adequate for reliability work, can be gained from the expression:

$$F(z) \approx \frac{1}{1+e^{-kz}} \quad \text{where } k = \sqrt{\frac{8}{\pi}} \text{ and } z = \frac{x-\mu}{\sigma}$$

5.2 Application

5.2.1 The Normal distribution describes the variability in many production processes and engineering characteristics, e.g. resistor, capacitor and inductance values, dimensional values, etc. It is sometimes used to describe the distribution of times to wear out.

5.2.2 If a random variable, x, is the result of summing many (usually 10 or more is recommended) other random variables, that are not highly inter-dependent, then it can be shown that the distribution of x approximates to the normal distribution. This result is known as the **Central Limit Theorem**.

5.3 Details

* Also (more recently) known as the Gaussian or LaPlace-Gauss distribution.

PDF: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (-\infty < x < +\infty)$

CDF: No analytical expression (but see 5.1.2 to 5.1.4).

mean: μ median: μ

mode: μ

variance: σ^2 standard deviation: σ

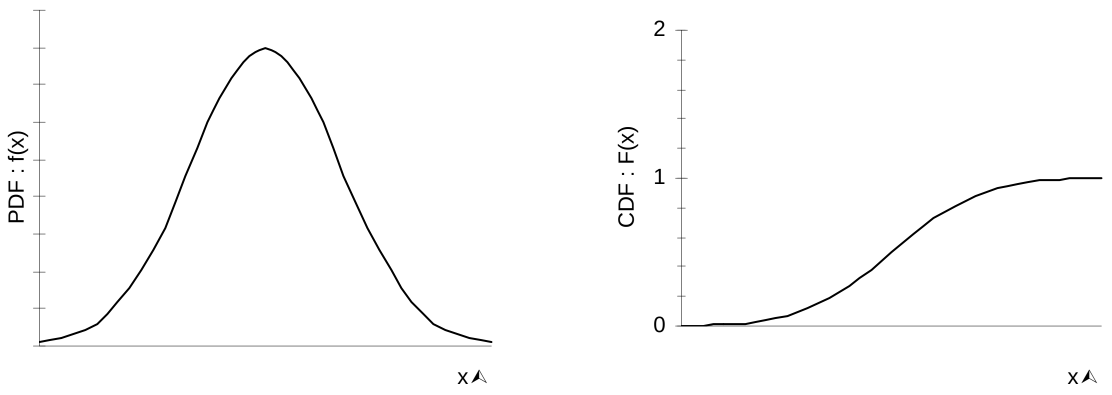

Figure 5: PDF and CDF of a normal distribution

6 LOG-NORMAL DISTRIBUTION

6.1 Nature

6.1.1 If the natural logarithm of the variable x [ln(x)] is normally distributed, then x forms a log-normal distribution (x>0). The log-normal distribution is defined by the parameters x_{50} (the median) and σ_e (the standard deviation of the natural logarithm of x).

6.2 Application

6.2.1 Many repair time distributions closely approximate to the Log-Normal distribution . It has been found empirically that, regardless of the value of x_m, for active repair times σ_e lies generally in the range 0.6 to 1.4, the lower values being associated with modular repair policies, the higher with detailed diagnosis and repair to component level. The Log-Normal distribution is also sometimes used to describe times to failure of equipment experiencing wear out.

6.3 Details

PDF: $$f(x) = \frac{1}{x\sigma_e\sqrt{2\pi}}.e^{-\frac{1}{2}\left(\frac{\ln(x/x_{50})}{\sigma_e}\right)^2}$$

CDF: No simple algebraic expression exists for F(x)

mean: $\mu = x_{50}.e^{\frac{\sigma_e^2}{2}}$

median: x_{50} is a parameter of the distribution

mode: $= x_{50}.e^{\sigma_e^2}$

variance: $= x_{50}^2.e^{\sigma_e^2}.\left(e^{\sigma_e^2} - 1\right)$

standard deviation: σ_e is a parameter of the distribution

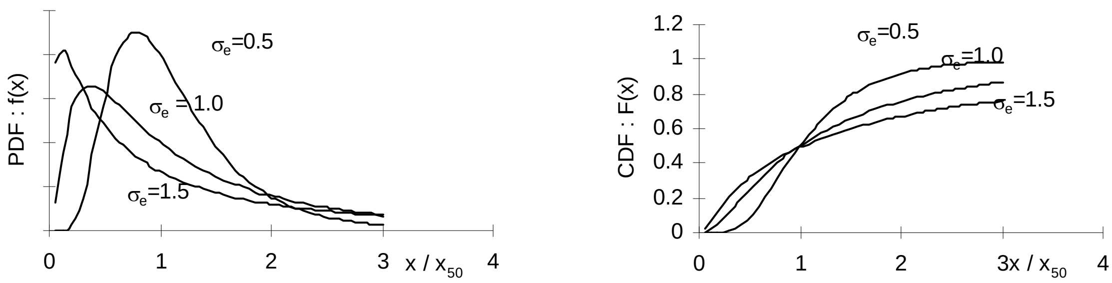

Figure 6: PDF and CDF of a log-normal distribution

7 CHI-SQUARED DISTRIBUTION

7.1 Nature

7.1.1 The χ^2 distribution is a distribution of a continuously random variable, χ^2, with a range from 0 to ∞, and a PDF as listed in section 7.3.

7.1.2 The sum of the squares of n independent standardised normal variables is a χ^2 random variable with parameter n. n is then called the **degrees of freedom**.

7.2 Application

7.2.1 The χ^2 distribution is used in goodness of fit tests and the derivation of confidence intervals. Both of these are addressed in Pt4 Ch7.

7.3 Details

PDF: $f(\chi^2,n) = \dfrac{\left(\chi^2\right)^{\left(\frac{n}{2}-1\right)}}{2^{\frac{n}{2}}.\Gamma\left(\frac{n}{2}\right)}.e^{-\frac{\chi^2}{2}}$ where the Gamma function $\Gamma(m)=\int_0^\infty e^{-x}.x^{m-1}.dx$

CDF: $F(\chi^2,n) = \dfrac{1}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)}.x.\int_0^x u^{\left(\frac{n}{2}-1\right)}.e^{-\frac{u}{2}}.du$

mean: $\mu = n$ variance: $\sigma = 2n$

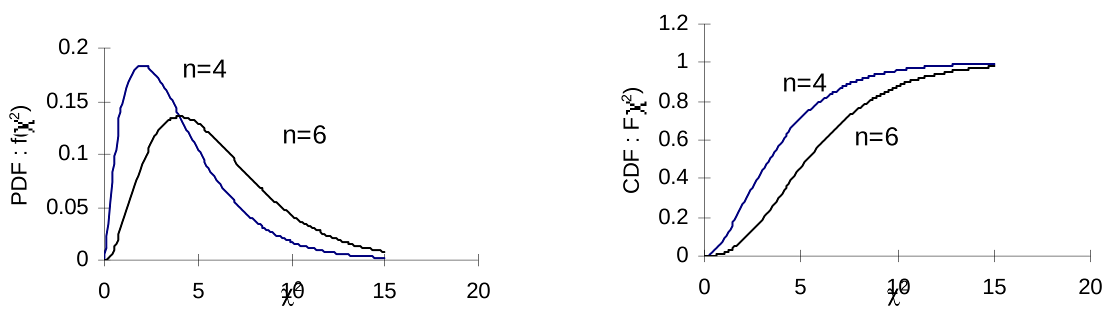

Figure 7: PDF and CDF for the χ^2 distribution

8 POISSON DISTRIBUTION

8.1 Nature

8.1.1 The Poisson Distribution is a discrete distribution in which x may take any positive integer value including zero. The PDF describes the probability of obtaining exactly x events when the expected (or mean) number of events is μ (μ need not be an integer).

8.2 Application

8.2.1 The Poisson Distribution can be applicable where the expected number of events in a given time is known and the total number in that time is unlimited.

8.2.2 For an item of equipment with constant failure rate λ, the Poisson Distribution models the potential number of failures in a given time, t. The expected number of failures, μ, is equal to λt. Inserting this value into the equation for PDF and solving for x = 0, 1, 2, etc provides a list of probabilities which relate to 0, 1, 2, etc failures occurring in time t. Such a model is useful in logistics planning where it will show how often there will be no repair work required and how often the planned resource will be overloaded.

8.3 Details

PDF: $f(x) = \frac{\mu^x}{x!} . e^{-x}$ for x = 0, 1, 2, 3, ..., ∞

CDF: $F(x) = \sum_{i=0}^{x} \frac{\mu i}{i!} . e^{-i}$ where i! = i.(i-1).(i-2). ... .(2).(1) and 0! = 1

mean: μ

mode: occurs when x is the largest integer less than μ

variance: μ

standard deviation: $\sigma = \sqrt{\mu}$

For $\mu > 9$ the Poisson distribution may be approximated by a Normal distribution with mean μ and variance μ.

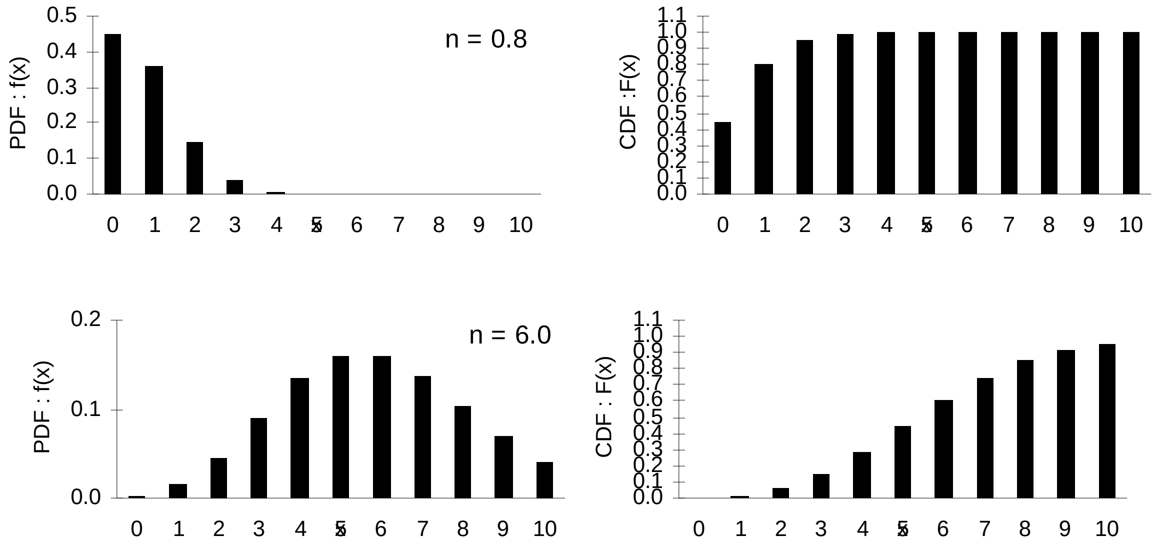

Figure 8: PDF and CDF for a Poisson Distribution

9 WEIBULL DISTRIBUTION

9.1 Nature

9.1.1 The Weibull distribution is a continuous distribution of wide generality. The Negative Exponential distribution is a special case of the Weibull.

9.2 Application

9.2.1 The Weibull distribution is of wide applicability in failure rate ($\lambda(t)$) analysis. It copes with the analysis of situations in which the failure rate is not constant. In the standard notation:

$$\lambda(t) = \frac{\beta . t^{(\beta-1)}}{\eta^{\beta}}$$

and cumulative (cumulative $\lambda(t)$ means average failure rate between 0 and t)

$$\lambda(t) = \left(\frac{t}{\eta}\right)^{\beta}$$

The survival function (R(t)) is also a simple expression, namely:

$$R(t) = e^{-\left(\frac{t}{\eta}\right)^{\beta}}$$

When $\beta < 1$, $\lambda(t)$ decreases with time and thus it can be used to describe the first part of the bath-tub curve of failure rate (see??).

When $\beta = 1$, $\lambda(t)$ = a constant (λ) and the Weibull distribution reduces to the Negative Exponential with $\lambda = 1/\eta$, or η = MTBF.

When $\beta > 1$, $\lambda(t)$ increases with time and it can be used to model the wear-out phase of the bath-tub curve.

When $\beta = 3.44$, the Weibull distribution approximates to the Normal distribution with

$$\mu = 0.9\eta$$

and $$\sigma^2 = 0.0835\eta^2$$

Weibull probability paper has been produced for plotting failure data to enable easy estimation of the parameters η and β.

9.3 Details

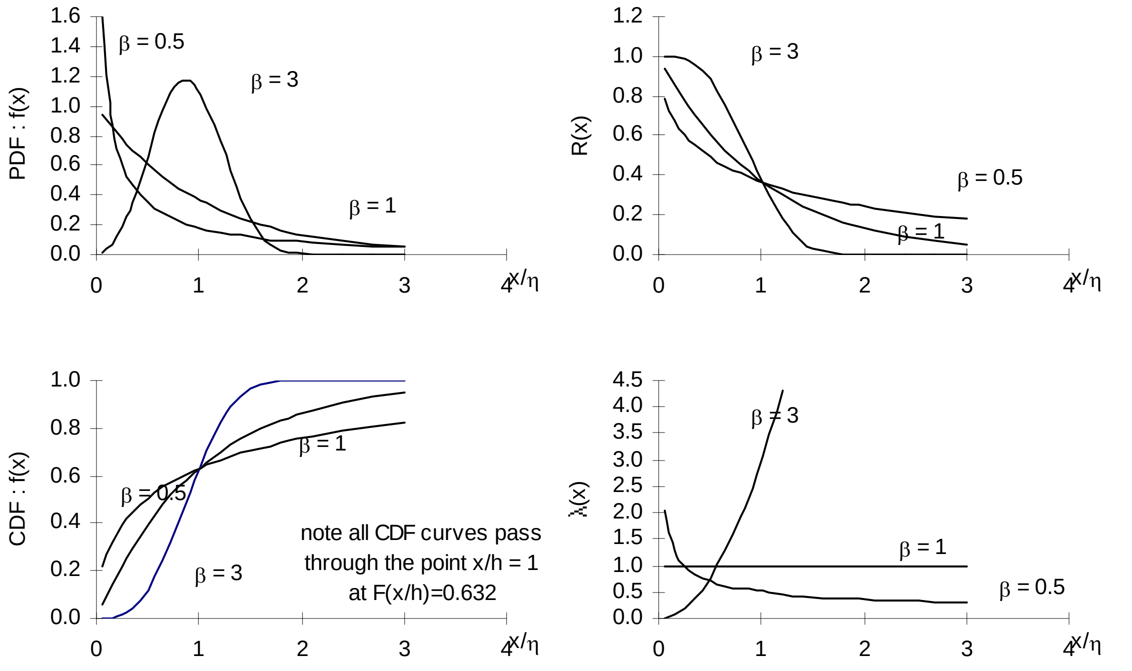

Figure 9: PDF, CDF, Reliability and Hazard Functions for a Weibull Distribution

PDF: $f(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{(\beta-1)}.e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}\right)}$

CDF: $F(x) = 1 - e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}\right)}$

where β is a constant known as the shape parameter,

η is a constant known as the scale parameter† ($\eta>0$),

$x \geq 0$

mean: $\mu = \eta.\Gamma\left(\frac{\beta+1}{\beta}\right)$ where Γ is the Gamma Function

mode: $\eta.\left(1-\frac{1}{\beta}\right)^{\frac{1}{\beta}}$ when $\beta>1$

† When t is time to failure, η is also known as the characteristic life, and measures the time by which 63.2% ($1-e^{-1}$) of items can be expected to have failed.

0 when $\beta \leq 1$

variance: $\sigma^2 = \eta^2 . \left(\Gamma\left(\frac{\beta+2}{\beta}\right) - \left(\Gamma\left(\frac{\beta+1}{\beta}\right)\right)^2 \right)$

standard deviation: σ

A location parameter (γ) may be necessary in practice; that is, x-γ may be Weibully distributed. In reliability work γ is sometimes known as the minimum life or failure-free life.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.00	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.10	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.20	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.30	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.40	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.50	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.60	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.70	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.80	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.90	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.00	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.10	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.20	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.30	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.40	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.50	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.60	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.70	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.80	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.90	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.00	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.10	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.20	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.30	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.40	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.50	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.60	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.70	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.80	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.90	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.00	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990

The table provides values for the area under the PDF between 0 and x. The area between $-\infty$ and 0 is 0.5. For negative x use symmetry.

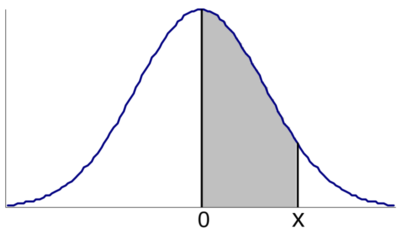

Table 1: Areas under the standard normal distribution

n \ γ	0.995	0.975	0.950	0.900	0.800	0.600	0.400	0.200	0.100	0.050	0.025	0.005
1	0.000039	0.000982	0.00393	0.0158	0.064	0.275	0.708	1.642	2.706	3.841	5.024	7.879
2	0.0100	0.0506	0.103	0.211	0.446	1.022	1.833	3.219	4.605	5.991	7.378	10.597
3	0.0717	0.216	0.352	0.584	1.005	1.869	2.946	4.642	6.251	7.815	9.348	12.838
4	0.207	0.484	0.711	1.064	1.649	2.753	4.045	5.989	7.779	9.488	11.143	14.860
5	0.412	0.831	1.145	1.610	2.343	3.656	5.132	7.289	9.236	11.070	12.832	16.750
6	0.676	1.237	1.635	2.204	3.070	4.570	6.211	8.558	10.645	12.592	14.449	18.548
7	0.989	1.690	2.167	2.833	3.822	5.493	7.283	9.803	12.017	14.067	16.013	20.278
8	1.344	2.180	2.733	3.490	4.594	6.423	8.351	11.030	13.362	15.507	17.535	21.955
9	1.735	2.700	3.325	4.168	5.380	7.357	9.414	12.242	14.684	16.919	19.023	23.589
10	2.156	3.247	3.940	4.865	6.179	8.295	10.473	13.442	15.987	18.307	20.483	25.188
11	2.603	3.816	4.575	5.578	6.989	9.237	11.530	14.631	17.275	19.675	21.920	26.757
12	3.074	4.404	5.226	6.304	7.807	10.182	12.584	15.812	18.549	21.026	23.337	28.300
13	3.565	5.009	5.892	7.041	8.634	11.129	13.636	16.985	19.812	22.362	24.736	29.819
14	4.075	5.629	6.571	7.790	9.467	12.078	14.685	18.151	21.064	23.685	26.119	31.319
15	4.601	6.262	7.261	8.547	10.307	13.030	15.733	19.311	22.307	24.996	27.488	32.801
16	5.142	6.908	7.962	9.312	11.152	13.983	16.780	20.465	23.542	26.296	28.845	34.267
17	5.697	7.564	8.672	10.085	12.002	14.937	17.824	21.615	24.769	27.587	30.191	35.718
18	6.265	8.231	9.390	10.865	12.857	15.893	18.868	22.760	25.989	28.869	31.526	37.156
19	6.844	8.907	10.117	11.651	13.716	16.850	19.910	23.900	27.204	30.144	32.852	38.582
20	7.434	9.591	10.851	12.443	14.578	17.809	20.951	25.038	28.412	31.410	34.170	39.997
21	8.034	10.283	11.591	13.240	15.445	18.768	21.992	26.171	29.615	32.671	35.479	41.401
22	8.643	10.982	12.338	14.041	16.314	19.729	23.031	27.301	30.813	33.924	36.781	42.796
23	9.260	11.689	13.091	14.848	17.187	20.690	24.069	28.429	32.007	35.172	38.076	44.181
24	9.886	12.401	13.848	15.659	18.062	21.652	25.106	29.553	33.196	36.415	39.364	45.558
25	10.520	13.120	14.611	16.473	18.940	22.616	26.143	30.675	34.382	37.652	40.646	46.928
26	11.160	13.844	15.379	17.292	19.820	23.579	27.179	31.795	35.563	38.885	41.923	48.290
27	11.808	14.573	16.151	18.114	20.703	24.544	28.214	32.912	36.741	40.113	43.195	49.645
28	12.461	15.308	16.928	18.939	21.588	25.509	29.249	34.027	37.916	41.337	44.461	50.994
29	13.121	16.047	17.708	19.768	22.475	26.475	30.283	35.139	39.087	42.557	45.722	52.335
30	13.787	16.791	18.493	20.599	23.364	27.442	31.316	36.250	40.256	43.773	46.979	53.672
35	17.192	20.569	22.465	24.797	27.836	32.282	36.475	41.778	46.059	49.802	53.203	60.275
40	20.707	24.433	26.509	29.051	32.345	37.134	41.622	47.269	51.805	55.758	59.342	66.766
45	24.311	28.366	30.612	33.350	36.884	41.995	46.761	52.729	57.505	61.656	65.410	73.166
50	27.991	32.357	34.764	37.689	41.449	46.864	51.892	58.164	63.167	67.505	71.420	79.490
55	31.735	36.398	38.958	42.060	46.036	51.739	57.016	63.577	68.796	73.311	77.380	85.749
60	35.534	40.482	43.188	46.459	50.641	56.620	62.135	68.972	74.397	79.082	83.298	91.952
65	39.383	44.603	47.450	50.883	55.262	61.506	67.249	74.351	79.973	84.821	89.177	98.105
70	43.275	48.758	51.739	55.329	59.898	66.396	72.358	79.715	85.527	90.531	95.023	104.215
75	47.206	52.942	56.054	59.795	64.547	71.290	77.464	85.066	91.061	96.217	100.839	110.285
80	51.172	57.153	60.391	64.278	69.207	76.188	82.566	90.405	96.578	101.879	106.629	116.321
85	55.170	61.389	64.749	68.777	73.878	81.089	87.665	95.734	102.079	107.522	112.393	122.324
90	59.196	65.647	69.126	73.291	78.558	85.993	92.761	101.054	107.565	113.145	118.136	128.299
95	63.250	69.925	73.520	77.818	83.248	90.899	97.855	106.364	113.038	118.752	123.858	134.247
100	67.328	74.222	77.929	82.358	87.945	95.808	102.946	111.667	118.498	124.342	129.561	140.170
105	71.428	78.536	82.354	86.909	92.650	100.719	108.035	116.962	123.947	129.918	135.247	146.069
110	75.550	82.867	86.792	91.471	97.362	105.632	113.121	122.250	129.385	135.480	140.916	151.948
120	83.852	91.573	95.705	100.624	106.806	115.465	123.289	132.806	140.233	146.567	152.211	163.648
130	92.223	100.331	104.662	109.811	116.272	125.304	133.450	143.340	151.045	157.610	163.453	175.278
140	100.655	109.137	113.659	119.029	125.758	135.149	143.604	153.854	161.827	168.613	174.648	186.847
150	109.142	117.985	122.692	128.275	135.263	145.000	153.753	164.349	172.581	179.581	185.800	198.360

$\chi^2_{\gamma,n}$ is the value of χ^2 with n degrees of freedom, exceeded with probability γ.

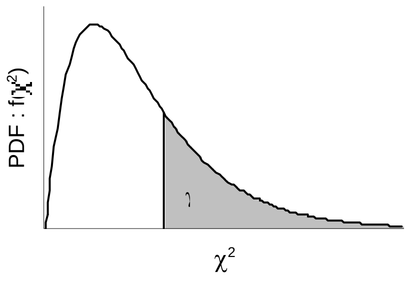

Table 2: Percentage points of the χ^2 distribution

Intentional blank page

LEAFLET 3/0

OTHER RELEVANT LITERATURE

1. BSi Standards: BS ISO 3534-1, *Statistics - Vocabulary and symbols* -, BSi 1993

Intentional blank page